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NODAL AND ASYMPTOTIC PROPERTIES OF SOLUTIONS TO NONLINEAR ELLIPTIC EIGENVALUE PROBLEMS ON GENERAL LEVEL SETS

BY

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ABSTRACT

We establish some nodal and asymptotic properties of the solutions of nonlinear elliptic eigenvalue problems obtained by Ljusternik–Schnirelman theory on general level sets.

1. Introduction

We consider the following nonlinear eigenvalue problem:

$$(1.1)\qquad \begin{cases} -\Delta u - c(x)u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$.

The purpose of this paper is to establish nodal and asymptotic properties of solutions to (1.1) obtained by Ljusternik–Schnirelman theory on the general level set

$$N_\alpha := \left\{ u \in \mathring{W}^{1,2}(\Omega);\ \frac{1}{2}\int_\Omega (|\nabla u|^2 - c(x)u^2)\,dx = \alpha, \right.$$

$$\left. \alpha < 0;\ \text{normalizing parameter} \right\}.$$

We impose the following conditions on f and c:

(A.1) $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is continuous.

(A.2) $f(x,-u) = -f(x,u), uf(x,u) \geq 0$ for $(x,u) \in \Omega \times \mathbf{R}$.

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(A.3) $0 < f(x,u)u - q(x)u^2 \le C|u|^{p+1}$ for some $C > 0$, $p > 1$, $0 < q(x) \in L^\infty(\Omega)$ and for all $(x,u) \in \Omega \times \{\mathbf{R}\setminus\{0\}\}$.

(A.4) $0 \le c(x) \in L^\infty(\Omega)$ and $c(x)u^2 \le C\int_0^u f(x,s)\,ds$ for some $C > 0$.

(A.5) $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_{n_0} < 0 < \lambda_{n_0+1} < \cdots$ for some $n_0 \in \mathbf{N}$, where λ_k is the k-th eigenvalue of the following linear eigenvalue problem:

$$\begin{cases} -\Delta u - c(x)u = \lambda q(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

(A.6) $f(x,u)/u$ is strictly increasing for $u > 0$ and every $x \in \Omega$.

Under the assumptions (A.1) ~ (A.5) Zeidler [3] showed that there exist solutions $(u_n(\alpha), \lambda_n(\alpha)) \in \mathring{W}^{1,2}(\Omega) \times \mathbf{R}$ ($\alpha < 0$, $1 \le n \le n_0$) which satisfy

$$\begin{cases} u_n(\alpha) \in N_\alpha, \\ \Psi(u_n(\alpha)) = \beta_n(\alpha) := \inf\limits_{K \in \Lambda_{n,\alpha}} \sup\limits_{u \in K} \Psi(u), \end{cases} \tag{1.3}$$

where

$$\Psi(u) := \int_\Omega dx \int_0^u f(x,s)\,ds \qquad \text{for } u \in \mathring{W}^{1,2}(\Omega),$$

$$\Lambda_{n,\alpha} := \{K \subset N_\alpha\text{: compact, symmetric with respect to the origin, } 0 \notin K \text{ and } \gamma(K) \ge n\},$$

where $\gamma(K)$ is the genus of K, which is defined in the next section. Such solutions are called *variational solutions* of (1.1).

It is easy to see that by (A.5), the level set N_α is not a sphere-like set but, roughly speaking, has the structure of a hyperboloid. Therefore, it seems meaningful to investigate the qualitative properties of such variational solutions on the general level set N_α.

First of all, we study the asymptotic properties of $u_n(\alpha)$ and $\lambda_n(\alpha)$ as $\alpha \uparrow 0$.

THEOREM 1. *Let* $1 \le n \le n_0$. *Assume* (A.1) ~ (A.5). *Furthermore, let* $1 < p < (N+2)/(N-2)$ $(N \ge 3)$, $1 < p < \infty$ $(N \le 2)$ *in* (A.3). *Then*

(a) *There exists a constant* $C > 0$ *such that for all* $-1 < \alpha < 0$

$$|\lambda_n(\alpha) - \lambda_n| \le C(\sqrt{-\alpha})^{p-1}.$$

(b) *There exist a sequence* $\{\alpha_j\}$ $(1 \le j)$ *such that* $\alpha_j \uparrow 0$ *and* $u_n(\alpha_j)/\sqrt{-\alpha_j} \to v_n$ *strongly in* $\mathring{W}^{1,2}(\Omega)$ *as* $j \to \infty$, *where* v_n *is the n-th eigenfunction of* (1.2) *satisfying*

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega (|\nabla v_n|^2 - c(x)v_n^2)\,dx = -1, \\ \Phi(v_n) = \beta_n := \inf\limits_{K \in \Lambda_{n,-1}} \sup\limits_{u \in K} \Phi(u), \end{cases} \tag{1.4}$$

where

$$\Phi(u) := \frac{1}{2}\int_\Omega q(x)u^2 dx \qquad \textit{for } u \in L^2(\Omega).$$

Next, we restrict our attention to the one dimensional case. Let $\Omega = (a,b)$ be an open bounded interval. In this case, it is well known that the n-th eigenfunction of (1.2) has exactly $n-1$ distinct interior zeroes (cf. Courant-Hilbert [1]). Then Theorem 1 suggests that $u_n(\alpha)$ has also exactly $n-1$ distinct interior zeroes. Our result for this problem is as follows:

THEOREM 2. *Assume* (A.1) ~ (A.6). *Let* $1 \le n \le n_0$. *Then there exists a variational solution* $(u_n(\alpha), \lambda_n(\alpha))$ *of* (1.1), *where* $u_n(\alpha)$ *has precisely* $n-1$ *distinct interior zeroes.*

REMARKS. (i) Consider the following nonlinear Sturm-Liouville problem:

$$\begin{cases} -u'' + f(x,u) = \lambda r(x) u & \text{in } (a,b), \quad r(x) > 0, \\ u(a) = u(b) = 0. \end{cases} \tag{1.5}$$

Heinz [2] showed the existence of a solution (u,λ) of (1.5) where u has exactly $n-1$ distinct interior zeroes. Such a solution was obtained by using Ljusternik-Schnirelman theory on the level set

$$S_\alpha := \left\{ u \in \mathring{W}^{1,2}(a,b); \int_a^b r(x)u^2 dx = \alpha \right\},$$

where α is a *positive* L^2-normalizing parameter.

(ii) The continuity of $u_n(\alpha)$ and $\lambda_n(\alpha)$ with respect to α seems to be unknown except the special cases.

In section 2 we give the proof of Theorem 1. Theorem 2 is proved in section 3.

2. Proof of Theorem 1

We explain notations and definitions. Let $X := \mathring{W}^{1,2}(\Omega)$ be the usual Sobolev space. For a given closed, symmetric (w.r.t. the origin) subset $K \subset X$ in which 0 is not contained, the genus of K, denoted by $\gamma(K)$, is defined by

$$\gamma(K) := \inf\{ n \in \mathbf{N}; \text{ there exists } h: K \to \mathbf{R}^n \setminus \{0\}, \ h \text{ is continuous and odd}\}.$$

In this section we fix the integer n $(1 \le n \le n_0)$.

LEMMA 2.1. *The eigenvalue and the associated eigenfunction of* (1.2) *obtained under the constraints* (1.4) *coincide with the n-th eigenvalue and the associated eigenfunction of* (1.2). *Furthermore,* $\beta_n = -1/\lambda_n$.

We can show this lemma easily by mathematical induction. Hence we omit the proof.

LEMMA 2.2. *For all* $-1 < \alpha < 0$, *there exists a constant* $C > 0$ *such that*

$$\int_\Omega |\nabla u_n(\alpha)|^2 dx \le C(-\alpha). \tag{2.1}$$

PROOF. Since $u_n(\alpha) \in N_\alpha$, we obtain from (A.4)

$$\int_\Omega |\nabla u_n(\alpha)|^2 dx = \int c(x) u_n(\alpha)^2 dx + 2\alpha$$

$$\le \int_\Omega dx \int_0^{u_n(\alpha)} f(x,s)\, ds + 2\alpha = \beta_n(\alpha) + 2\alpha.$$

Therefore, we have only to show that $\beta_n(\alpha) \le C(-\alpha)$. By (1.3) we can choose $K \subset \Lambda_{n,-1}$ and by (A.3), compactness of K and Sobolev's embedding theorem we obtain

$$\beta_n(\alpha) \le \sup_{v\in K} \int_\Omega dx \int_0^{\sqrt{-\alpha}v} f(x,s)\, ds \le C \sup_{v \in K} \int_\Omega dx \int_0^{|\sqrt{-\alpha}v|} (|s|^p + q(x)|s|)\, ds$$

$$\le C \sup_{v\in K} \int_\Omega \{(\sqrt{-\alpha})^{p+1} |v|^{p+1} + (-\alpha) q(x) v^2\}\, dx$$

$$\le C\{(\sqrt{-\alpha})^{p+1} + (-\alpha)\} \le C(-\alpha).$$

Thus we get (2.1). q.e.d.

LEMMA 2.3. *There exists a constant* $C > 0$ *such that for any* $-1 < \alpha < 0$

$$|1/\lambda_n(\alpha) - \beta_n(\alpha)/\alpha| \le C(\sqrt{-\alpha})^{p-1}. \tag{2.2}$$

PROOF. Since $(u_n(\alpha), \lambda_n(\alpha))$ is the solution of (1.1) and $u_n(\alpha) \in N_\alpha$, we have by integration by parts

$$2\alpha = \int_\Omega (|\nabla u_n(\alpha)|^2 - c(x) u_n(\alpha)^2)\, dx = \lambda_n(\alpha) \int_\Omega f(x, u_n(\alpha)) u_n(\alpha)\, dx. \tag{2.3}$$

We obtain by (A.3), (1.3), (2.3), Sobolev's embedding theorem and Lemma 2.2 that

$$\begin{aligned}|\alpha/\lambda_n(\alpha) - \beta_n(\alpha)| &= \left|\frac{1}{2}\int_\Omega f(x,u_n(\alpha))u_n(\alpha)\,dx - \int_\Omega dx \int_0^{u_n(\alpha)} f(x,s)\,ds\right| \\ &\leq \frac{1}{2}\int_\Omega |f(x,u_n(\alpha)) - q(x)u_n(\alpha)|\,|u_n(\alpha)|\,dx \\ &\quad + \int_\Omega dx \int_0^{u_n(\alpha)} |f(x,s) - q(x)s|\,ds \\ &\leq C\int_\Omega |u_n(\alpha)|^{p+1}\,dx \leq C(\sqrt{-\alpha})^{p+1}.\end{aligned}$$

Thus we get (2.2). q.e.d.

LEMMA 2.4. *There exists a constant $C > 0$ such that for any $-1 < \alpha < 0$*

$$|\beta_n(\alpha)/\alpha - 1/\lambda_n| \leq C(\sqrt{-\alpha})^{p-1}. \tag{2.4}$$

PROOF. By (A.3) we have for $u \in \mathbf{R}$

$$\int_0^u q(x)s\,ds \leq \int_0^u f(x,s)\,ds \leq \int_0^u q(x)s\,ds + C|u|^{p+1}. \tag{2.5}$$

We integrate (2.5) with respect to x to obtain

$$\frac{1}{2}\int_\Omega q(x)u^2\,dx \leq \Psi(u) \leq \frac{1}{2}\int_\Omega q(x)u^2\,dx + C\int_\Omega |u|^{p+1}\,dx. \tag{2.6}$$

Let $K \in \Lambda_{n,\alpha}$. Then it follows from (2.6) that

$$\begin{aligned}\sup_{u\in K}\frac{1}{2}\int_\Omega q(x)u^2\,dx &\leq \sup_{u\in K}\Psi(u) \\ &\leq \sup_{u\in K}\frac{1}{2}\int_\Omega q(x)u^2\,dx + \sup_{u\in K} C\int_\Omega |u|^{p+1}\,dx.\end{aligned} \tag{2.7}$$

We note that there exists $K_0 \in \Lambda_{n,\alpha}$ such that

$$\inf_{K\in\Lambda_{n,\alpha}}\sup_{u\in K}\frac{1}{2}\int_\Omega q(x)u^2\,dx = \sup_{u\in K_0}\frac{1}{2}\int_\Omega q(x)u^2\,dx. \tag{2.8}$$

In fact, $K_0 = \operatorname{span}\{u_1, u_2, \ldots, u_{n-1}, u_n\} \cap N_\alpha$, where u_j $(1 \le j \le n)$ is the j-th eigenfunction of (1.2). By using (2.7) and (2.8) we obtain

$$-\alpha\beta_n \le \beta_n(\alpha) \le \inf_{K \in \Lambda_{n,\alpha}} \left\{ \sup_{u \in K} \frac{1}{2} \int_\Omega q(x) u^2 dx + \sup_{u \in K} C \int_\Omega |u|^{p+1} dx \right\}$$

$$\le \sup_{u \in K_0} \frac{1}{2} \int_\Omega q(x) u^2 dx + \sup_{u \in K_0} C \int_\Omega |u|^{p+1} dx$$

$$= -\alpha\beta_n + \sup_{u \in K_0} C \int_\Omega |u|^{p+1} dx,$$

from which it follows that

$$|\beta_n(\alpha) + \beta_n\alpha| \le \sup_{u \in K_0} C \int_\Omega |u|^{p+1} dx. \tag{2.9}$$

Define $K_1 := \{v \in X;\ \sqrt{-\alpha}\, v \in K_0\}$. Since $\beta_n = -1/\lambda_n$, by using (2.9), Sobolev's embedding theorem, we get

$$|1/\lambda_n - \beta_n(\alpha)/\alpha| \le \sup_{v \in K_1} C \int_\Omega (\sqrt{-\alpha})^{p-1} |v|^{p+1} dx \le C(\sqrt{-\alpha})^{p-1}.$$

Thus the proof is complete. q.e.d.

Consequently, combining Lemma 2.3 and 2.4 we get Theorem 1(a).

Next we shall prove Theorem 1(b) by using Theorem 1(a). For this purpose we prepare some fundamental lemmas. We put $v_\alpha = u_n(\alpha)/(\sqrt{-\alpha})$.

Lemma 2.5. *$v_\alpha \to v_0$ $(\neq 0)$ weakly in X as $\alpha \uparrow 0$, where v_0 is the n-th eigenfunction of* (1.2).

Proof. By Lemma 2.2, $\{v_\alpha\}$ is bounded in X. Hence, we can choose a weakly convergent subsequence. Let v_0 be the weak limit of $\{v_\alpha\}$. By Sobolev's embedding theorem $\{v_\alpha\}$ converges to v_0 strongly in $L^{p+1}(\Omega)$ and $L^2(\Omega)$. From (1.1) we see that

$$-\Delta v_\alpha - c(x) v_\alpha = \lambda_n(\alpha) f(x, \sqrt{-\alpha}\, v_\alpha)/\sqrt{-\alpha}. \tag{2.10}$$

Multiplying $w \in X$ on both sides of (2.10) and integrating by parts, we obtain

$$\int_\Omega (\nabla v_\alpha \cdot \nabla w - c(x) v_\alpha w)\, dx = \lambda_n(\alpha) \int_\Omega f(x, \sqrt{-\alpha}\, v_\alpha) w/\sqrt{-\alpha}\, dx. \tag{2.11}$$

Then it is easy to see that the left-hand side of (2.11) converges to $\int_\Omega (\nabla v_0 \cdot \nabla w - c(x) v_0 w)\, dx$ as $\alpha \uparrow 0$. By using (A.3), Hölder's inequality, Sobolev's embedding theorem and Lemma 2.2 we obtain

$$\left| \int_\Omega f(x, \sqrt{-\alpha} v_\alpha) w / \sqrt{-\alpha}\, dx - \int_\Omega q(x) v_\alpha w\, dx \right| \leq C(\sqrt{-\alpha})^{p-1} \int_\Omega |v_\alpha|^p |w|\, dx$$

$$\leq C(\sqrt{-\alpha})^{p-1} \left(\int_\Omega |v_\alpha|^{p+1} dx \right)^{p/(p+1)} \left(\int_\Omega |w|^{p+1} dx \right)^{1/(p+1)} \to 0 \qquad \text{as } \alpha \uparrow 0.$$

Hence, by letting $\alpha \uparrow 0$ in (2.11), we obtain from Theorem 1(a)

$$\int_\Omega \nabla v_0 \cdot \nabla w\, dx - \int_\Omega c(x) v_0 w\, dx = \lambda_n \int_\Omega q(x) v_0 w\, dx \qquad \text{for } w \in X, \tag{2.12}$$

which implies that v_0 is a weak solution of (1.2) with respect to $\lambda = \lambda_n$. Finally, we show that $v_0 \neq 0$. Assume $v_0 = 0$. Then noting that $v_\alpha \in N_{-1}$, we have for sufficiently small α

$$\int_\Omega |\nabla v_\alpha|^2 dx = -2 + \int_\Omega c(x) v_\alpha^2 dx < 0.$$

This is a contradiction. Hence, we obtain $v_0 \neq 0$. q.e.d.

LEMMA 2.6. *The following equality holds*:

$$\frac{1}{2} \int_\Omega q(x) v_0^2\, dx = -1/\lambda_n. \tag{2.13}$$

PROOF. From (1.1) we have

$$2\alpha = \int_\Omega |\nabla u_n(\alpha)|^2 dx - \int_\Omega c(x) u_n(\alpha)^2 dx = \lambda_n(\alpha) \int_\Omega f(x, u_n(\alpha)) u_n(\alpha)\, dx,$$

from which, with Theorem 1(a), we obtain as $\alpha \uparrow 0$

$$\int_\Omega f(x, \sqrt{-\alpha} v_\alpha) v_\alpha / \sqrt{-\alpha}\, dx = -2/\lambda_n(\alpha) \to -2/\lambda_n. \tag{2.14}$$

By Lemma 2.2, 2.5 and (A.3) we obtain

$$\begin{aligned} &\left| \int_\Omega (f(x, \sqrt{-\alpha} v_\alpha) v_\alpha / \sqrt{-\alpha} - q(x) v_\alpha^2)\, dx \right| \\ &\qquad \leq C(\sqrt{-\alpha})^{p-1} \int_\Omega |v_\alpha|^{p+1} dx \leq C(\sqrt{-\alpha})^{p-1}. \end{aligned} \tag{2.15}$$

Since $v_\alpha \to v_0$ strongly in $L^2(\Omega)$, we get (2.13) by (2.14) and (2.15). q.e.d.

LEMMA 2.7. $v_0 \in N_{-1}$.

PROOF. Put $w = v_0$ in (2.11). By a similar calculation as that in (2.15), we obtain from Lemma 2.6 and Theorem 1(a), by letting $\alpha \uparrow 0$,

$$\int_\Omega |\nabla v_0|^2 dx - \int_\Omega c(x) v_0^2 dx = \lambda_n \int_\Omega q(x) v_0^2 dx = -2.$$

Therefore, we get $v_0 \in N_{-1}$. q.e.d.

LEMMA 2.8. *$v_\alpha \to v_0$ strongly in X as $\alpha \uparrow 0$.*

PROOF. Since $v_\alpha \in N_{-1}$, we obtain by Lemma 2.7

$$\int_\Omega |\nabla v_\alpha|^2 dx = -2 + \int_\Omega c(x) v_\alpha^2 dx \to -2 + \int_\Omega c(x) v_0^2 dx = \int_\Omega |\nabla v_0|^2 dx. \tag{2.16}$$

Noting that v_0 is the weak limit of v_α in X, we immediately get our conclusion by (2.16). q.e.d.

As a consequence of Lemmas 2.5 and 2.8, we get Theorem 1(b).

3. Proof of Theorem 2

In this section we consider the one dimensional ODE case:

$$-u'' - c(x)u = \lambda f(x,u) \qquad \text{in } \Omega = (a,b) \quad (-\infty < a < b < \infty), \tag{3.1}$$

$$u(a) = u(b) = 0. \tag{3.2}$$

We prove Theorem 2 by using the idea of Heinz [2]. We begin with the preparation of a fundamental lemma.

LEMMA 3.1. *Let (u,λ) and (u,μ) satisfy* (3.1) *in $J = (c,d)$. Assume that $\lambda, \mu < 0$ and $u, v > 0$ in J. Furthermore, assume that $u(c) = v(c) = y_0$, $u(d) = v(d) = y_1$. Then*

(i) *If $u \le v$ in J, then $\lambda \le \mu$.*
(ii) *If $u \le v$ in J and $\lambda = \mu$, then $u \equiv v$ in J.*

PROOF. We put $B := u'v - uv'$. We easily obtain $u'(c) \le v'(c)$ and $v'(d) \le u'(d)$. Then we have

$$B(d) = (u'(d) - v'(d))y_1 \ge 0, \qquad B(c) = (u'(c) - v'(c))y_0 \le 0.$$

By using (1.1) we obtain

$$
\begin{aligned}
0 \le B(d) - B(c) &= \int_c^d (dB/dx)\,dx = \int_c^d (u''v - uv'')\,dx \\
&= \int_c^d \{-\lambda f(x,u)/u + \mu f(x,v)/v\} uv\,dx.
\end{aligned}
\tag{3.3}
$$

We put $G(x) := -\lambda f(x,u)/u + \mu f(x,v)/v$. Assume that $G(x)$ changes sign in J. Then there exists $\sigma \in J$ such that $G(\sigma) = 0$, i.e. $\lambda f(\sigma, u(\sigma))/u(\sigma) = \mu f(\sigma, v(\sigma))/v(\sigma)$. Then by (A.6) we obtain $\lambda \le \mu$. Next, assume that $G(x) \ge 0$ in J. Then by (A.6) we obtain

$$-\lambda \ge -\mu(f(x,v)/v)/(f(x,u)/u) \ge -\mu.$$

Hence the proof of (i) is complete.

We prove (ii). From (3.3) and $\lambda = \mu$, we have

$$0 \le -\lambda \int_c^d \{f(x,u)/u - f(x,v)/v\} uv\,dx.$$

Noting that $\lambda < 0$, by (A.6) we obtain $u \equiv v$ in J. q.e.d.

LEMMA 3.2. *Let* (u,λ) *and* (v,μ) *satisfy* (1.1) *in* $J = (c,d)$. *Suppose that* $\lambda, \mu < 0$ *and* $u, v > 0$ *in* J. *Then*

(i) $\lambda = \mu$ *if and only if* $u = v$ *in* J.

(ii) $\lambda < \mu$ *if and only if* $u < v$ *in* J.

PROOF. If $u = v$, then $\lambda = \mu$ is trivial. Suppose that $\lambda = \mu$. Let there exist $x_0 \in J$ such that $u(x_0) < v(x_0)$. We put

$$\beta := \sup\{x; u(x) < v(x)\}, \qquad \nu := \inf\{x; u(x) < v(x)\}.$$

Since $u(\nu) = v(\nu) \ge 0$, $u(\beta) = v(\beta) \ge 0$ and $u(x) \le v(x)$ in (ν, β) we can apply Lemma 3.1(ii) to get $u \equiv v$ in (ν, β), which is a contradiction. Hence, the proof of (i) is complete.

We prove (ii). If $u < v$ in J, then by Lemma 3.1(i) we know $\lambda \le \mu$. Moreover, if $\lambda = \mu$, then by Lemma 3.1(ii) we obtain $u \equiv v$ in J. Hence, we get $\lambda < \mu$. Next, assume that $\lambda < \mu$. Then we can easily see that $h := v - u \ge 0$ in J. If there exists $x_0 \in (c,d)$ such that $h(x_0) = 0$, then $\sigma := u(x_0) = v(x_0) > 0$ and h attains a local minimum at $x = x_0$. Therefore, we obtain

$$0 \le h''(x_0) = v''(x_0) - u''(x_0) = -\mu f(x_0,\sigma) + \lambda f(x_0,\sigma)$$
$$= f(x_0,\sigma)(-\mu + \lambda) < 0,$$

which is a contradiction. Hence, $u < v$ in J. q.e.d.

We consider the auxiliary variational problem which depends on $\alpha < 0$ and a finite partition $a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b$ of Ω, and it reads as follows:

(3.4) Minimize Ψ under the constraints $u \in N_\alpha$, $u(x_k) = 0$ $(0 \le k \le n)$.

We shall use the following notation. Let $Z := \{x_1, \ldots, x_{n-1}\}$ $(x_j \le x_{j+1})$, Z_n be the union of such Z. $J_k := (x_{k-1}, x_k)$ $(0 \le k \le n)$ and

$$B(\alpha, Z) := \{u \in N_\alpha; u(t) = 0 \text{ for any } t \in Z\}.$$

Furthermore, we denote by $Q(\alpha, Z)$ the set of solutions of the problem (3.4).

Problem (3.4) is deeply connected with the piecewise solution of (3.1) and (3.2). Precisely, we call (u,λ) the piecewise solution of (3.1) and (3.2) with respect to α and Z if $(u,\lambda) \in B(\alpha, Z) \times \mathbf{R}$ and u is a classical solution of (3.1) and (3.2) on each J_k of the partition given by Z. Denote by $P(\alpha, Z)$ the set of the piecewise solutions with respect to α and Z. Finally, for $u \in X$, the zero of u is called a *nodal zero* iff it is interior to Ω but not interior to the set of all zeroes of u. Denote by $N(u)$ the set of all nodal zeroes of u.

LEMMA 3.3. *Let $(u,\lambda), (v,\mu) \in Q(\alpha, Z)$ such that $N(u), N(v) \subset Z$. Put*

$$I_1 := \{x \in \Omega; |u(x)| \le |v(x)|\},$$
$$I_2 := \{x \in \Omega; |u(x)| > |v(x)|\}.$$

Then

(a) *For any k, either $J_k \subset I_1$ or $J_k \subset I_2$.*
(b) *If $\lambda \le \mu < 0$, then $v \equiv 0$ in I_2.*
(c) *If $\lambda \le \mu < 0$, then $I_2 = \varnothing$.*

PROOF. (a) can be proved by a similar method to that used in the proof of [2, Lemma 3.3]. Hence we omit the proof. (b) follows from Lemma 3.3(ii). We shall show (c). We assume that $I_2 \ne \varnothing$ and derive a contradiction. For this purpose, we construct a differential curve $(u_\xi)_{0 \le \xi \le 1}$ in $B(\alpha, Z)$ such that $u_0 = v$ and

(3.5) $$d\Psi(u_\xi)/d\xi|_{\xi=0} < 0.$$

Then we see that (3.5) contradicts the fact that $u \in Q(\alpha, Z)$. Put

$$u_\xi = \begin{cases} (1-\xi)v & \text{in } I_1, \\ \rho(\xi)^{1/2}u & \text{in } I_2, \end{cases}$$

where $\rho(\xi)$ will be specified later. We put

$$\nu := \int_{I_2} (|u'|^2 - c(x)u^2)\, dx.$$

By (b) we see that $v \equiv 0$ in I_2. Then we have

$$\int_\Omega (|u_\xi'|^2 - c(x)u_\xi^2)\, dx = (1-\xi)^2 2\alpha + \rho(\xi)\nu.$$

Suppose that $\nu = 0$. Then

$$\int_\Omega (|u'|^2 - c(x)u^2)\, dx = \int_{I_1} (|u'|^2 - c(x)u^2)\, dx = 2\alpha.$$

Put

$$w := \begin{cases} u & \text{in } I_1, \\ 0 & \text{in } I_2. \end{cases}$$

Since $I_2 \neq \varnothing$ and $w \in B(\alpha, Z)$, we easily see that $\Psi(w) < \Psi(u) = \beta$, which is a contradiction. Hence we obtain $\nu \neq 0$. Next, suppose that $\nu > 0$. We put

$$2\alpha_0 := \int_{I_1} (|u'|^2 - c(x)u^2)\, dx = 2\alpha - \nu < 2\alpha < 0.$$

We define

$$\tilde{w} := \begin{cases} \sqrt{-\alpha}u/\sqrt{-\alpha_0} & \text{in } I_1, \\ 0 & \text{in } I_2. \end{cases}$$

It is obvious that $\tilde{w} \in B(\alpha, Z)$ and $|\tilde{w}| < |u|$ in I_1. Hence, we obtain $\Psi(\tilde{w}) < \Psi(u) = \beta$, which is a contradiction. Hence, we obtain $\nu < 0$. Now, we define

$$\rho(\xi) := 2\alpha(2\xi - \xi^2)/\nu \geq 0 \qquad \text{for } \xi \in [0,1],$$

$$\Psi_i(u) := \int_{I_i} dx \int_0^u f(x,s)\, ds \qquad \text{for } u \in X \quad \text{and} \quad i = 1,2.$$

By direct calculation we easily obtain

$$d\Psi_1(u_\xi)/d\xi\,|_{\xi=0} = -\int_{I_1} f(x, v(x))v(x)\,dx, \tag{3.6}$$

$$d\Psi_2(u_\xi)/d\xi\,|_{\xi=0} = 2\alpha/\nu \int_{I_2} q(x)u^2\,dx. \tag{3.7}$$

Since $v \equiv 0$ in I_2, we obtain

$$2\alpha = \int_\Omega (|v'|^2 - c(x)v^2)\,dx = \mu \int_\Omega f(x,v)v\,dx = \mu \int_{I_1} f(x,v)v\,dx,$$

from which we have

$$\int_{I_1} f(x,v)v\,dx = 2\alpha/\mu. \tag{3.8}$$

By integration by parts, we obtain

$$\nu = \int_{I_2} (|u'|^2 - c(x)u^2)\,dx = \lambda \int_{I_2} f(x,u)u\,dx. \tag{3.9}$$

It follows from (A.3) and (3.9) that

$$\int_{I_2} q(x)u^2\,dx < \int_{I_2} f(x,u)u\,dx = \nu/\lambda. \tag{3.10}$$

Combining (3.6), (3.7), (3.8) and (3.10) we obtain

$$d\Psi(u_\xi)/d\xi\,|_{\xi=0} = -2\alpha/\mu + 2\alpha/\nu \int_{I_2} q(x)u^2\,dx < -2\alpha/\mu + 2\alpha/\lambda \le 0.$$

Hence we get (3.5). q.e.d.

LEMMA 3.4. *Let $\alpha < 0$ and Z be fixed. Put $\beta := \inf_{u\in B(\alpha,Z)} \Psi(u)$. Then*

(a) $\beta > 0$.

(b) *There exists $u_0 \in B(\alpha, Z)$ uniquely such that $\Psi(u_0) = \beta$ and $u_0 \ge 0$ in Ω.*

(c) *$Q(\alpha, Z)$ consists of all continuous functions u such that $|u| = u_0$.*

(d) *There exists $\lambda_0 < 0$ such that $(u, \lambda_0) \in P(\alpha, Z)$ for any $u \in Q(\alpha, Z)$.*

PROOF. Since $\psi(u) \ge 0$ for any $u \in X$, it is obvious that $\beta \ge 0$. Suppose that $\beta = 0$. We can choose minimizing sequence $\{u_j\} \subset B(\alpha, Z)$ such that $\beta = \lim_{j\to\infty} \Psi(u_j)$. By using (A.4) we obtain

$$\int_\Omega |u_j'|^2\,dx = \int_\Omega c(x)u_j^2\,dx + 2\alpha \le \int_\Omega dx \int_0^{u_j} f(x,s)\,ds + 2\alpha = \Psi(u_j) + 2\alpha.$$

Since $\Psi(u_j) \to 0$, for sufficiently large j, we see from the inequality just above that $\int_\Omega |u_j'|^2\,dx < 0$, which is a contradiction. Hence we get Lemma 3.3(a).

From (a) we know that $\{u_j\}$ is bounded in X. Hence, we can choose a weakly convergent subsequence of $\{u_j\}$, which is written $\{u_j\}$ again. Let u be the weak limit of $\{u_j\}$. We show that $u \in B(\alpha, Z)$. By Sobolev's embedding theorem, we easily obtain that $u(t_j) = 0$ for $t_j \in Z$. By lower semicontinuity of the norm of X, we have

$$\int_\Omega |u'|^2\,dx \le \liminf_{j\to\infty} \int_\Omega |u_j'|^2\,dx. \tag{3.11}$$

Since $u_j \to u$ strongly in $L^2(\Omega)$, we obtain

$$\int_\Omega c(x)u^2\,dx = \lim_{j\to\infty} \int_\Omega c(x)u_j^2\,dx. \tag{3.12}$$

By (3.11) and (3.12) we obtain

$$2\alpha_0 := \int_\Omega |u'|^2\,dx - \int_\Omega c(x)u^2\,dx \le \liminf_{j\to\infty}\left(\int_\Omega (|u_j'|^2 - c(x)u_j^2\right)dx = 2\alpha.$$

Assume $\alpha_0 < \alpha$. We put $v := \sqrt{-\alpha}u/\sqrt{-\alpha_0} \in B(\alpha, Z)$. Since $\sqrt{-\alpha}/\sqrt{-\alpha_0} < 1$, we obtain that $|v| < |u|$ in Ω_0, where Ω_0 is the non-empty subset of Ω in which $u \ne 0$. Therefore, by $u_j \to u$ strongly in $L^{p+1}(\Omega)$, we obtain

$$\Psi(v) = \int_\Omega dx \int_0^v f(x,s)\,ds < \int_\Omega dx \int_0^u f(x,s)\,ds = \lim_{j\to\infty} \int_\Omega dx \int_0^{u_j} f(x,s)\,ds = \beta,$$

which contradicts the definition of β. Hence, $\alpha_0 = \alpha$, i.e., $u \in N_\alpha$. Therefore, we get $u \in N_\alpha$ and $\Psi(u) = \beta$, namely, $u \in Q(\alpha, Z)$.

Let $(u,\lambda), (v,\mu) \in Q(\alpha, Z)$ be such that $u, v \ge 0$ in Ω. Without loss of generality, we may assume that $\lambda \le \mu < 0$. By Lemma 3.3(c) we obtain that $0 \le u \le v$ in Ω. Since $\Psi(u) = \Psi(v) = \beta$, we obtain $u \equiv v$ in Ω. Hence, we get (b).

(c) is an immediate consequence of (b). We can prove (d) by using the Lagrange Multiplier Theorem. q.e.d.

We now investigate the continuity of the elements of $Q(\alpha, Z)$ with respect to Z. Let u_0 be the unique nonnegative element of $Q(\alpha, Z)$. For $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{+1, -1\}^n$ we put $U_n(\alpha, Z, \sigma) := \sigma_k u_0(x)$ for $x \in J_k$.

LEMMA 3.5. *Let* $Z_j := \{t_{jk}; 1 \le k \le n\}$ *satisfy* $t_{0k} = \lim_{j\to\infty} t_{jk}$ *for* $1 \le k \le n$. *Then* $U_n(\alpha, Z_j, \sigma) \to U_n(\alpha, Z_0, \sigma)$ *strongly in* X.

PROOF. We put $u_j := U_n(\alpha, Z_j, \sigma)$, $\beta_j := \Psi(u_j)$. We easily obtain

$$\overline{\lim_{j\to\infty}}\, \beta_j \le \beta_0 \tag{3.13}$$

by using the method used in the proof of [2, Lemma 3.6]. Then it follows from (A.4) and (3.13) that

$$\int_\Omega |u_j'|^2\, dx = 2\alpha + \int_\Omega c(x) u^2\, dx \le 2\alpha + C\Psi(u_j) = 2\alpha + C\beta_j \le 2\alpha + C\beta_0.$$

Hence, we may assume that $u_j \to v$ weakly in X. By Sobolev's embedding theorem we easily see that $v(t_{0k}) = 0$ for any $t_{0k} \in Z_0$. We put

$$2\alpha_0 := \int_\Omega (|v'|^2 - c(x)v^2)\, dx.$$

It follows from weakly lower semicontinuity of the norm of X that $\alpha_0 \le \alpha$. Assume $\alpha_0 < \alpha$. Then $\sqrt{-\alpha}v/\sqrt{-\alpha_0} \in B(\alpha, Z_0)$ and we obtain

$$\Psi(\sqrt{-\alpha}v/\sqrt{-\alpha_0}) < \Psi(v) = \lim_{j\to\infty} \Psi(u_j) \le \overline{\lim_{j\to\infty}}\, \beta_j \le \beta_0,$$

which is a contradiction. Hence, we get $\alpha_0 = \alpha$, namely, $v \in Q(\alpha, Z_0)$. Finally, we can easily show that $v = U_n(\alpha, Z_0, \sigma)$ by using Sobolev's embedding theorem. Hence the proof is complete. q.e.d.

Now we prove the following Theorem 3.6. Theorem 2 is the immediate consequence of Theorem 3.6(e).

Let $Z_n := \{Z = \{a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b\};\ m \le n\}$ and $P_n := \bigcup_{Z\in Z_n} Q(\alpha, Z)$. For $c \in \mathbf{R}$, K_c denotes the set of all $u \in N_\alpha \cap \Psi^{-1}(c)$ which are critical points of the restriction of Ψ to N_α.

THEOREM 3.6. *Assume* (A.1) ~ (A.6). *Then for any* $1 \le n \le n_0$

(a) P_n *is compact, symmetric,* $0 \notin P_n$ *and* $\gamma(P_n) = n$.

(b) $P_n \cap K_{\beta_n(\alpha)} \ne \emptyset$.

(c) $\beta_n(\alpha) = \max_{u\in P_n} \Psi(u)$.

(d) $\beta_n(\alpha) < \beta_{n+1}(\alpha)$.

(e) *If* $u \in P_n \cap K_{\beta_n(\alpha)}$, *then* u *has precisely* $n - 1$ *zeroes in* Ω.

Proof. We first prove that $\gamma(P_n) \geq n$ and P_n is compact. We put $S := \{h \in \mathbf{R}^n; \sum_{j=1}^n |h_j| = 1\}$. For $h \in S$, we define $Z(h) := \{t_1(h), \ldots, t_{n-1}(h)\}$, where $t_k(h) := a + (b-a)\sum_{j=1}^k |h_j|$. For $\sigma \in \{+1,-1\}^n$, put

$$\Theta_\sigma(h) := U_n(\alpha, Z(h), \sigma) \in P_n,$$

$$D_\sigma := \{h \in S; \sigma_j h_j \geq 0, j = 1, \ldots, n\}.$$

Then by Lemma 3.5 we see that the mapping $\Theta_\sigma : D_\sigma \to X$ is continuous and odd. Furthermore, we define $\Theta : S \to X$ by $\Theta|_{D_\sigma} = \Theta_\sigma$. Then it is easily seen that Θ is continuous, odd and $\Theta(S) = P_n$ by Lemma 3.4(c). Hence, we obtain from Borsuk–Ulam's Theorem, namely, $\gamma(S) = n$, that

$$n = \gamma(S) \leq \gamma(\Theta(S)) = \gamma(P_n).$$

Since Θ is continuous and S is compact in $\mathbf{R}^n$, P_n is compact.

Assume that $P_n \cap K_{\beta_n(\alpha)} = \varnothing$. Since P_n is compact, there exists an open set $W \supset K_{\beta_n(\alpha)}$ in N_α such that $W \cap P_n = \varnothing$. We know from [3, Proposition 1] that, under the conditions (A.1) ~ (A.6), there exists a deformation $d : N_\alpha \times [0,1] \to N_\alpha$ and $\epsilon > 0$ such that the map $u \mapsto d(u,t)$ is odd in u and a homeomorphism from N_α onto N_α for any $t \in [0,1]$. Moreover,

$$\Psi(u) \leq \beta_n(\alpha) + \epsilon \quad \text{and} \quad u \in N_\alpha \backslash W \quad \text{imply} \quad \Psi(d(u,1)) \leq \beta_n(\alpha) - \epsilon.$$

Hence we obtain

$$\gamma(P_n) \leq \gamma(d(P_n,1)) \leq \gamma(N_\alpha \cap \Psi^{-1}(-\infty, \beta_n(\alpha) - \epsilon)) < n,$$

which is a contradiction. Therefore the proof of (b) is complete.

We put $b_n := \max_{u \in P_n} \Psi(u)$. Since $\gamma(P_n) \geq n$, it follows from the definition of $\beta_n(\alpha)$ that

$$\beta_n(\alpha) \leq \max_{u \in P_n} \Psi(u) = b_n.$$

On the other hand, for any $\epsilon > 0$, there exists $A \in \Lambda_{n,\alpha}$ such that

$$\sup_{u \in A} \Psi(u) \leq \beta_n(\alpha) + \epsilon. \tag{3.14}$$

Since P_n is compact, there exists $w \in P_n$ such that $\Psi(w) = b_n$. Let $Z = \{t_1, \ldots, t_{m-1}\}$ $(m \leq n)$ be the partition such that $w \in Q(\alpha, Z)$. We define $E : X \to \mathbf{R}^{m-1}$ by

$$E(u) := (u(t_1), \ldots, u(t_{m-1})).$$

If $A \cap B(\alpha,Z) = \varnothing$, then we obtain $\gamma(A) \le m-1$, which contradicts $A \in \Lambda_{n,\alpha}$. Hence, $A \cap B(\alpha,Z) \ne \varnothing$. Let $v \in A \cap B(\alpha,Z)$. Then we obtain by $w \in Q(\alpha,Z)$ that

$$b_n = \Psi(w) \le \Psi(v) \le \sup_{u\in A} \Psi(u) \le \beta_n(\alpha) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get $b_n \le \beta_n(\alpha)$. Hence the proof of (c) is complete.

Let $v_1 \in P_n \cap K_{\beta_n(\alpha)}$ and $v_1 \in Q(\alpha,Z_1)$. For $s \in \Omega \backslash Z_1$, we put $Z_2 := Z_1 \cup \{s\}$. It is clear that $u(s) \ne 0$. Let $v_2 \in Q(\alpha,Z_2)$. Then it is obvious that $v_2 \in B(\alpha,Z_1)$ but, by Lemma 3.4(c), $v_2 \in Q(\alpha,Z_1)$, since $v_2(s) = 0$. Therefore, we obtain by (c) that

$$\beta_n(\alpha) = \psi(v_1) < \Psi(v_2) \le \max_{u\in P_{n+1}} \Psi(u) = \beta_{n+1}(\alpha).$$

Hence the proof of (d) is complete.

Suppose that $\gamma(P_n) \ge n+1$. Then by definition of $\beta_{n+1}(\alpha)$ and (c) we have

$$\beta_{n+1}(\alpha) \le \sup_{u\in P_n} \Psi(u) = \beta_n(\alpha),$$

which contradicts (d). Hence the proof of (a) is complete.

Finally, let $u \in P_n \cap K_{\beta_n(\alpha)}$ and m be the number of zeroes of u in Ω. Since $u \in P_n$, we obtain $m \le n-1$. On the other hand, since $u \in P_{m+1}$, by using (c) we obtain

$$\beta_n(\alpha) = \Psi(u) \le \max_{v\in P_{m+1}} \Psi(v) = \beta_{m+1}(\alpha).$$

Then from (d) we obtain $n \le m+1$. Hence we get $m = n-1$, which is the conclusion of (e). q.e.d.

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